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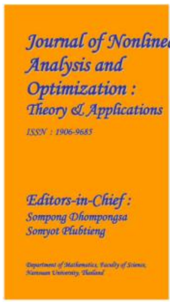

ADVANCEMENTS IN COMPUTATIONAL TECHNIQUES FOR SOLVING NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The Natural Transform Method (NTM) and its applications are the main topic of this paper's exploration of recent developments in computing methods for solving nonlinear ordinary differential equations (ODEs). With regard to solving nonlinear ODEs, the paper presents the Natural Decomposition Method (NDM) as a novel approach and gives a thorough explanation of the NTM, including its integral transforms like the Laplace and Sumudu transforms. Complex differential equations can have exact or approximate solutions derived by the NDM using NTM and a series solution framework that includes recursive relations and Adomian polynomials. Worked examples showing that solutions obtained with the NDM closely match known outcomes highlight the usefulness of these methods. Researchers and industry professionals can benefit greatly from these developments in computational approaches, which offer notable gains in the efficiency and accuracy of solving nonlinear ODEs.

Keywords: Computational, Techniques, Solving, Nonlinear, Ordinary Differential Equations

1. INTRODUCTION

The comprehension and simulation of intricate behaviors in diverse scientific and engineering domains, including as physics, biology, chemistry, economics, and engineering, heavily relies on the study of nonlinear ordinary differential equations, or ODEs. Nonlinear ODEs, in contrast to linear differential equations, include complex interactions in which the dependent variable and its derivatives manifest in nonlinear ways. Complex dynamics like chaos, bifurcations, and other phenomena that linear systems are unable to explain are frequently the result of this nonlinearity. Because of this, it is frequently impractical to solve nonlinear ODEs analytically, which calls for the development of sophisticated computational methods in order to produce approximate solutions with a reasonable level of accuracy. Nonlinear ODEs have traditionally been solved using conventional numerical techniques including the Euler method, Runge-Kutta methods, and finite difference approaches. Even though these techniques have been successful in resolving a wide range of issues, their use is frequently constrained by problems with accuracy, stability, and convergence-particularly when working with stiff equations or highly nonlinear systems. Due to these constraints, computational mathematics has seen a great deal of study and invention, which has resulted in the creation of increasingly complex methods that can manage the complexity of nonlinear ODEs. With recent developments, a new paradigm for solving nonlinear ODEs has been made possible by machine learning-based techniques, mainly neural networks. These methods make use of neural networks' data-driven architecture to learn intricate patterns and correlations from training data in order to approximate solutions.

Besides, to further develop arrangement precision and registering productivity, metaheuristic calculations like Differential Advancement (DE), Molecule Multitude Improvement (PSO), and Hereditary Calculations (GA) have been joined with traditional mathematical techniques. These halves and half methodologies give stable arrangements even in circumstances including complex limit conditions or multi-faceted nonlinear frameworks by consolidating the upsides of both deterministic and stochastic methodologies. The development of adaptive and multiscale approaches has also been prompted by the increasing demand for accuracy and efficiency in solving nonlinear ODEs. These methods guarantee that the numerical solutions are both computationally feasible and correct by dynamically adjusting discretisation and computational resources according to the local features of the issue. Furthermore, approaches like as finite element and spectral methods have been improved to better manage the difficulties presented by stiff equations and the complexity of nonlinearities.

2. REVIEW OF LITREATURE

Ahmad et al. (2020) investigate new viewpoints on regular answers for nonlinear time fragmentary halfway differential equations, underscoring the job of partial math in working on the exactness and productivity of mathematical arrangements. Their review offers inventive methodologies for handling nonlinearities by incorporating partial request administrators, which give a more extensive system to demonstrating processes with memory impacts. This work exhibits how fragmentary math based strategies can be really applied to a large number of nonlinear frameworks, offering better intermingling and security contrasted with conventional techniques.

Arora and Ram (2024) provide a comprehensive overview of advanced numerical techniques for solving both linear and nonlinear differential equations. Their edited volume consolidates research on modern computational strategies, ranging from iterative methods to sophisticated hybrid approaches. The book accentuates the significance of choosing proper techniques in light of the idea of the differential equations being tackled, offering bits of knowledge into the compromises between computational intricacy and arrangement precision. The creators examine different mathematical plans, including limited contrast, limited component, and otherworldly strategies, while additionally investigating the job of AI and advancement calculations in improving the exhibition of these techniques. This work fills in as a significant asset for the two specialists and experts looking to figure out the most recent progressions in mathematical techniques for differential equations.

Bhatti et al. (2020) focus on the recent trends in computational fluid dynamics (CFD) and their application in solving nonlinear differential equations. Their research highlights the growing use of advanced numerical methods and simulation techniques to model complex fluid flows governed by nonlinear ODEs and PDEs. The authors discuss the integration of CFD with modern computational tools, such as machine learning and data-driven approaches, which have significantly enhanced the predictive capabilities of CFD models. By addressing challenges related to stability, convergence, and accuracy, this work underscores the importance of leveraging hybrid methods that combine deterministic and stochastic techniques for solving nonlinear fluid dynamics problems.

3. FUNDAMENTAL THOUGHT OF THE NORMAL CHANGE TECHNIQUE

We give some foundation data with respect to the idea of the Normal Change Strategy (NTM) in this part. Given a capability $f(t)$, where $t \in (-\infty, \infty)$, the overall basic change has the accompanying definition:

$$\Im[f(t)](s) = \int_{-\infty}^{\infty} K(s,t)f(t)dt, \ldots (1) \quad \mathcal{I}[f(t)](s) = \int_{-\infty}^{\infty} K(s,t)f(t)dt$$

where $K(s,t)$ means the change's piece and s is a genuine, complex number that is free of t. Remember that tJn(st) and t when K(s, t) is e⁻s t^{s-1} (st), then Equation (1) gives the Hankel change, Mellin change, and Laplace change, in a specific order change.

Let us now investigate the integral transformations described by for $(t), t \in (-\infty, \infty)$:

$$\begin{aligned} \Im[f(t)](u) &= \int_{-\infty}^{\infty} K(t)f(ut)dt \\ \Im[f(t)](s,u) &= \int_{-\infty}^{\infty} K(s,t)f(ut)dt \end{aligned} \quad \ldots (2)$$

It is vital to take note of that, when $\mathrm{K(t)} = \mathrm{e} - \mathrm{t}$, the fundamental Sumudu change is given by Equation (2), where u is utilized instead of boundarys. Moreover, the summed-up Laplace and Sumudu changes are characterized, individually for any worth of:

$$\begin{aligned} \ell[f(t)] &= F(s) = s^n \int_0 e^{-s^{n+1}t} f(s^n t)dt \\ \mathbb{S}[f(t)] &= G(u) = u^n \int_0^{\infty} e^{-u^n t} f(tu^{n+1})dt \end{aligned} \quad \ldots (3)$$

Observe that the Laplace and Sumudu changes, individually, are addressed by Equations (2) and (3) when n = 0.

4. Definitions and Properties of the N-Change

The capability $f(t)$ for $t \in (-\infty, \infty)$ has the accompanying regular change characterized by:

$$\mathbb{N}[f(t)] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut)dt; s,u \in (-\infty,\infty) \quad \text{.....(4)}$$

where the factors s and u are the regular change factors and $N[f(t)]$ is the time capability $f(t$'s) regular change. Remember that Eq. (4) can be communicated as:

$$\begin{aligned}\mathbb{N}[f(t)] &= \int_{-\infty}^{\infty} e^{-st} f(ut)dt; s,u \in (-\infty,\infty) \\ &= \left[\int_{-\infty}^{0} e^{-st} f(ut)dt; s,u \in (-\infty,0)\right] + \left[\int_{0}^{\infty} e^{-st} f(ut)dt; s,u \in (0,\infty)\right] \\ &= \mathbb{N}^{-}[f(t)] + \mathbb{N}^{+}[f(t)] \\ &= \mathbb{N}[f(t)H(-t)] + \mathbb{N}[f(t)H(t)] \\ &= R^{-}(s,u) + R^{+}(s,u)\end{aligned} \quad \text{..(5)}$$

The Heaviside capability is addressed by H(.). This ought to be noted: if the capability $f(t)H(t)$ is characterized on the positive genuine hub for all t values in R, then, at that point, the Regular change (N-Change) is characterized on the set

$$A = \left\{ f(t): \exists M, \tau_1, \tau_2 > 0, \text{ such that } |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0,\infty), j \in \mathbb{Z}^+ \right\} \quad \text{....(6)}$$

$$\mathbb{N}[f(t)H(t)] = \mathbb{N}^{+}[f(t)] = R^{+}(s,u) = \int_{0}^{\infty} e^{-st} f(ut)dt; s,u \in (0,\infty)$$

the Heaviside capability is addressed by H(.). Remember that Equation (6) can be diminished to the Sumudu change if s = 1 and to the Laplace change if u = 1. We currently give a portion of the N Changes alongside their transformation to Laplace and Sumudu.

Table 1: Unique N-Transforms and the transformation to Laplace and Sumudu

$f(t)$	$\mathbb{N}[f(t)]$	$\mathbb{S}[f(t)]$	$\ell[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n = 1,2,\dots$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Remark 1: Further information regarding the Natural transform can be found. Here is some crucial N -Transform qualities that we now present:

Table 2: Properties of N-Transforms

Functional Form	Natural Transform

$y(t)$	$Y(s,u)$
$y(at)$	$\frac{1}{a}Y(s,u)$
$y'(t)$	$\frac{s}{u}Y(s,u)-\frac{y(0)}{u}$
$y''(t)$	$\frac{s^2}{u^2}Y(s,u)-\frac{s}{u^2}y(0)-\frac{y'(0)}{u}$
$\gamma y(t)\pm\beta v(t)$	$\gamma Y(s,u)\pm\beta V(s,u)$

5. THE NORMAL DETERIORATION STRATEGY

We exhibit how the Normal Decay Technique might be utilized to settle a couple of nonlinear ordinary differential equations in this segment.

5.1 Methodology of the NDM:

Analyze the accompanying general nonlinear ordinary differential condition:

$$Lv+R(v)+F(v)=g(t)\ldots(7)$$

based on the original condition

$$v(0)=h(t)\ldots.(8)$$

where g(t) is the nonhomogeneous term, F(v) is the nonlinear term, R is the differential administrator's lingering, and L is the administrator of the greatest subordinate. Assuming L is a first-order differential operator, we can obtain the following by using the N − Transform of Equation (7):

$$\frac{sV(s,u)}{u}-\frac{V(0)}{u}+\mathbb{N}^{+}[R(v)]+\mathbb{N}^{+}[F(v)]=\mathbb{N}^{+}[g(t)]\ldots.(9)$$

When we change Eq. (8) to Eq. (9), we get:

$$V(s,u)=\frac{h(t)}{s}+\frac{u}{s}\mathbb{N}^{+}[g(t)]-\frac{u}{s}\mathbb{N}^{+}[R(v)+F(v)]\ldots.(10)$$

By taking the N -Change of Condition (10) contrarily, we acquire:

$$v(t)=G(t)-\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[R(v)+F(v)]\right]\ldots..(11)$$

where the source term is G(t). Now, we'll assume that the unknown function $v(t)$ of the following form has an infinite series solution:

$$v(t)=\sum_{n=0}^{\infty} v_n(t)\ldots.(12)$$

Next, we can rewrite Eq. (11) in the following form by applying Eq. (12):

$$\sum_{n=0}^{\infty} v_n(t)=G(t)-\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[R\sum_{n=0}^{\infty} v_n(t)+\sum_{n=0}^{\infty} A_n(t)]\right]\ldots.(13)$$

where the nonlinear term is represented by the Adomian polynomial An(t). We can quickly construct the recursive relation by comparing the two sides of Eq. (13) as shown below:

$$v_0(t) = G(t)$$
$$v_1(t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[Rv_0(t) + A_0(t)]\right]$$
$$v_2(t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[Rv_1(t) + A_1(t)]\right] \ldots.(14)$$
$$v_3(t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[Rv_2(t) + A_2(t)]\right]$$

We eventually have the broad recursive relation shown here:

$$v_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[Rv_n(t) + A_n(t)]\right],\ n \geq 0 \ldots.(15)$$

Therefore, the exact or approximative answer is provided by:

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \ldots.(16)$$

6. Examples

This section compares our solutions to the actual solutions that already exist after applying the NDM to three real-world scenarios.

Example 1: Look at the accompanying first-request nonlinear differential condition:

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dt}\right)^2 + v^2(t) = 1 - \sin(t) \ldots.(17)$$

based on the original condition

$$v(0) = 0,\ v'(0) = 1 \ldots(18)$$

To begin with, we apply the N -change to the two sides of Condition (5.1), and the outcome is:

$$\frac{s^2V(s,u)}{u^2} - \frac{sV(0)}{u^2} - \frac{v'(0)}{u} + \mathbb{N}^{+}\left[\left(\frac{dv}{dt}\right)^2\right] + \mathbb{N}^{+}[v^2(t)] = \frac{1}{s} - \frac{u}{s^2+u^2} \ldots(19)$$

We get the following by changing Eq. (18) to Eq. (19):

$$V(s,u) = \frac{u^2}{s^3} + \frac{u}{s^2+u^2} - \frac{u^2}{s^2}\mathbb{N}^{+}\left[\left(\frac{dv}{dt}\right)^2 + v^2(t)\right] \ldots.(20)$$

Next, using Eq. (20 inverse N-Transform), we obtain:

$$v(t) = \frac{t^2}{2!} + \sin(t) - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\left(\frac{dv}{dt}\right)^2 + v^2(t)\right]\right] \ldots.(21)$$

Now, we'll assume that the unknown function $v(t)$ has an infinite series solution of the following form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \ldots.(22)$$

We may rewrite Eq. (21) as follows by utilizing Eq. (22):

$$\sum_{n=0}^{\infty} v_n(t) = \frac{t^2}{2!} + \sin(t) - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right] \ldots.(23)$$

where the Adomian polynomials of the nonlinear terms (dv/dt)2 and $v^2(t)$ are, separately, A_n and B_n. From that point forward, we might drive the overall recursive connection as follows by contrasting the different sides of Equation (23):

$$v_0(t) = \frac{t^2}{2!} + \sin(t)$$
$$v_1(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[A_0 + B_0]\right]$$
$$v_2(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[A_1 + B_1]\right] \ldots\ldots(24)$$
$$v_3(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[A_2 + B_2]\right]$$

The broad recursive relation is thus provided by

$$v_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[A_n + B_n]\right],\ n \geq 0 \ldots.(25)$$

The excess parts of the obscure capability v(t) can subsequently be basically processed as follows utilizing the recursive connection characterized in Eq. (25):

$$v_1(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[A_0 + B_0]\right]$$
$$= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[(v_0')^2 + v_0^2]\right]$$
$$= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[(v_0')^2 + v_0^2]\right] \ldots..(26)$$
$$= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+[1]\right] + \cdots$$
$$= -\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] + \cdots$$
$$= -\frac{t^2}{2!} + \cdots$$

Thusly, it is feasible to exhibit that the non-dropped part of $v_0(t)$ still fulfills the gave differential condition by dropping the clamor terms that arise somewhere in the range of $v_0(t)$ and $v_1(t)$. This prompts a definite arrangement of the accompanying structure:

$$v(t) = \sin(t) \ldots.(27)$$

The precise answer closely matches the outcome that (ADM) was able to acquire.

$$\frac{dv}{dt} - 1 = v^2(t) \ldots\ldots(28)$$

subject to the initial condition

$$v(0) = 0 \ldots\ldots..(29)$$

Equation (29), when the Natural transform is applied to both sides, yields:

$$\frac{s}{u}V(s,u) - \frac{1}{u}V(s,u) - \frac{1}{s} = \mathbb{N}^+[v^2(t)] \ldots\ldots(30)$$

Replacing Equation (29)

$$V(s,u) = \frac{u}{s^2} + \frac{u}{s}\left[\mathbb{N}^+[v^2(t)]\right] \ldots..(31)$$

we attain

$$v(t) = t + \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^+[v^2(t)]\right]\right] \ldots.(32)$$

We now assume that the unknown function v(t) has an infinite solution of the following form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \ldots.(33)$$

We can rewrite Eq. (32) using Eq. (33) as follows:

$$\sum_{n=0}^{\infty} v_n(t) = t + \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} A_n(t)\right]\right]\right] \ldots..(34)$$

we can engender the recursive relative as trails:

$$\begin{aligned} v_0(t) &= t \\ v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_0(t)]\right]\right] \\ v_2(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_1(t)]\right]\right] \ldots..(35) \\ v_3(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_2(t)]\right]\right] \end{aligned}$$

Consequently, the following represents the general recursive relation:

$$v_{n+1}(t) = \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_n(t)]\right]\right],\ n \geq 0 \ldots \ldots (36)$$

We can quickly calculate the remaining elements of the unknown function v(t) using Eq. (36) as follows:

$$\begin{aligned} v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_0(t)]\right]\right] = \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[v_0^2(t)]\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[t^2]\right]\right] = \mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] = \frac{1}{3}t^3, \\ v_2(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_1(t)]\right]\right] = \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[2v_0(t)v_1(t)]\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{2t^4}{3}\right]\right]\right] = \mathbb{N}^{-1}\left[\frac{48u^5}{3s^6}\right] = \frac{2t^5}{15} \qquad \ldots..(37) \\ v_3(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_2(t)]\right]\right] = \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[2v_0(t)v_2(t) + v_1^2(t)]\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{17t^6}{45}\right]\right]\right] = \mathbb{N}^{-1}\left[\frac{12240u^7}{45s^8}\right] = \frac{17t^7}{315} \end{aligned}$$

Then the approximate explanation of the unidentified function $v(t)$ is specified by:

$$\begin{aligned} v(t) &= \sum_{n=0}^{\infty} v_n(t) \\ &= v_0(t) + v_1(t) + v_2(t) + v_3(t) + \cdots \\ &= t + \frac{1}{3}t^3 + \frac{2t^5}{15} + \frac{17t^7}{315} + \cdots \end{aligned}$$

Now we get

$$v(t) = \tan(t)$$

The exact solution is in closed agreement with the result obtained by (ADM).

Example 2: Examine the following first-order nonlinear ordinary differential equation:

$$\frac{dv}{dt} = 1 - t^2 + v^2(t) \ldots (38)$$

based on the original condition

$$v(0) = 0 \ldots .(39)$$

Applying the Natural transform to each side of Equation (29) yields the following result:

$$\frac{s}{u}V(s,u)-\frac{1}{u}V(s,u)-\frac{1}{s}=\mathbb{N}^{+}[v^{2}(t)]\ldots.(40)$$

By changing Eq. (40), we get:

$$V(s,u)=\frac{u}{s^{2}}+\frac{u}{s}\left[\mathbb{N}^{+}[v^{2}(t)]\right]\ldots.(41)$$

Using Eq. (41)'s inverse Natural transform, we get:

$$v(t)=t+\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[v^{2}(t)]\right]\right]\ldots.(42)$$

The unknown function $v(t)$ of the following form is now assumed to have an infinite solution:

$$v(t)=\sum_{n=0}^{\infty} v_{n}(t)\ldots.(43)$$

We may rewrite Eq. (42) as follows using Eq. (43):

$$\sum_{n=0}^{\infty} v_{n}(t)=t+\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} A_{n}(t)\right]\right]\right]\ldots..(44)$$

And the nonlinear term $V2(t)$ is represented by the Adomian polynomial An(t). Next, we can create the recursive relation as follows using Eq. (44):

$$\begin{aligned}
v_{0}(t)&=t\\
v_{1}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{0}(t)]\right]\right]\\
v_{2}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{1}(t)]\right]\right]\ldots..(45)\\
v_{3}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{2}(t)]\right]\right]
\end{aligned}$$

Consequently, the generic recursive relation is provided by:

$$v_{n+1}(t)=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{n}(t)]\right]\right],\ n\geq 0\ldots..(46)$$

The remaining components of the unknown function v(t) are simply computed as follows using Eq. (46):

$$\begin{aligned}
v_{1}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{0}(t)]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[v_{0}^{2}(t)]\right]\right]\\
&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[t^{2}]\right]\right]=\mathbb{N}^{-1}\left[\frac{2u^{3}}{s^{4}}\right]=\frac{1}{3}t^{3}\\
v_{2}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{1}(t)]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[2v_{0}(t)v_{1}(t)]\right]\right]\\
&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{2t^{4}}{3}\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{48u^{5}}{3s^{6}}\right]=\frac{2t^{5}}{15} \qquad \ldots..(47)\\
v_{3}(t)&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[A_{2}(t)]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}[2v_{0}(t)v_{2}(t)+v_{1}^{2}(t)]\right]\right]\\
&=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{17t^{6}}{45}\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{12240u^{7}}{45s^{8}}\right]=\frac{17t^{7}}{315}
\end{aligned}$$

Next, the following represents the approximative solution of the unknown function (t) :

$$\begin{aligned}
v(t)&=\sum_{n=0}^{\infty} v_{n}(t)\\
&=v_{0}(t)+v_{1}(t)+v_{2}(t)+v_{3}(t)+\cdots\ldots..(48)\\
&=t+\frac{1}{3}t^{3}+\frac{2t^{5}}{15}+\frac{17t^{7}}{315}+\cdots
\end{aligned}$$

Thus, the following provides the precise answer to Eq. (48):

$$v(t) = \tan(t) \ldots..(49)$$

The precise answer closely matches the outcome that (ADM) was able to acquire.

7. CONCLUSION

The Natural Transform Method (NTM) and related methodologies have tremendous promise, as demonstrated by the developments in computational strategies for solving nonlinear ordinary differential equations (ODEs). The NTM provides a strong foundation for solving nonlinear ODEs. It includes several integral transforms, including the Laplace, Sumudu, and their generalized variants. The Natural Decomposition Method (NDM), which incorporates recursive relations and Adomian polynomials, offers a methodical way to solve nonlinear differential equations through a series solution technique by utilizing these transforms. The given examples show how the NDM can be applied practically to get perfect or approximate answers that closely match known solutions. These developments make it a useful tool for researchers and practitioners in the field by streamlining the computational process and improving the accuracy of solutions to difficult differential equations.

8. REFERENCES

[1] H.Ahmad, A. Akgül, T. A. Khan, P. S. Stanimirović, Y. M. Chu, New perspective on the conventional solutions of the nonlinear time-fractional partial differential equations, Complexity 2020(1) (2020) 8829017.

[2] G.Arora, M. Ram (Eds.), Advance Numerical Techniques to Solve Linear and Nonlinear Differential Equations, CRC Press, 2024.

[3] M.M.Bhatti, M. Marin, A. Zeeshan, S. I. Abdelsalam, Recent trends in computational fluid dynamics, Frontiers in Physics 8 (2020) 593111.

[4] J.C.Butcher, Numerical Methods for Ordinary Differential Equations, Joh Wiley & Sons, 2016.

[5] L.Debnath, Nonlinear Partial Differential Equations for Scientists and Engineers, Birkhäuser, Boston, 2005, pp. 528–529.

[6] Q. Du, Nonlocal Modeling, Analysis, and Computation: Nonlocal Modeling, Analysis, and Computation, Society for Industrial and Applied Mathematics, 2019.

[7] R. Grimshaw, Nonlinear Ordinary Differential Equations, Routledge, 2017.

[8] D. Jordan, P. Smith, Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, OUP Oxford, 2007.

[9] M.M.Khater, Advancements in computational techniques for precise solitary wave solutions in the (1+1)-dimensional Mikhailov-Novikov-Wang equation, International Journal of Theoretical Physics 62(7) (2023) 152.

[10] A. Kumar, Control of Nonlinear Differential Algebraic Equation Systems with Applications to Chemical Processes, Chapman and Hall/CRC, 2020.

[11] A.Raza, M. Rafiq, N. Ahmed, M. S. Iqbal, S. Rezapour, M. Inc, Computer modeling: A gateway to novel advancements in solving real-life problems, Biomedical Signal Processing and Control 95 (2024) 106414.

[12] C.Yip, L. Seol, X. Z. Hon, A step-by-step approach to partial differential equations, Fusion of Multidisciplinary Research, An International Journal 3(1) (2022) 302–315.